The Online Metric Matching Problem for Doubling Metrics

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Outline

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The Problem

Setting:

- k servers are placed in a metric space
- k requests appear anywhere on the metric space, and each must be matched to an unassigned server.

Goal:

- minimize the cost of the matching

Assignment Rules:

- the requests come one at a time, and we must match each request up as soon as they arrive
- as soon as a request is matched to some server, it can never be reassigned

An Example

● = requests

■ = servers

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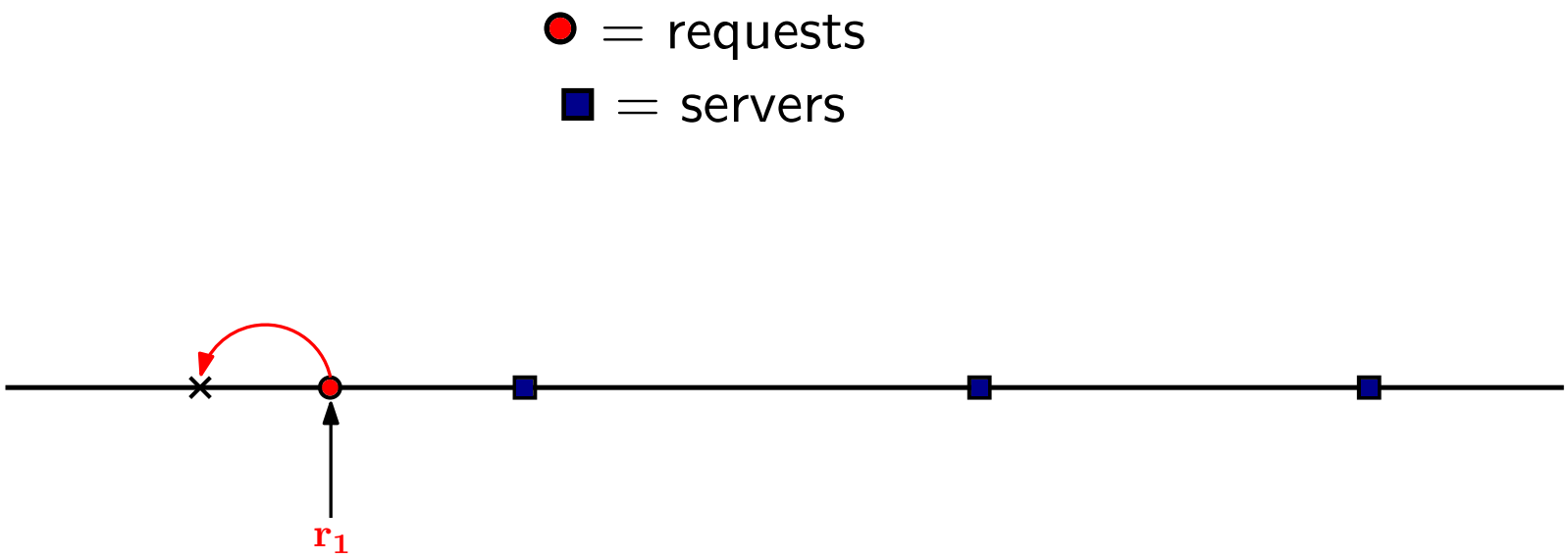

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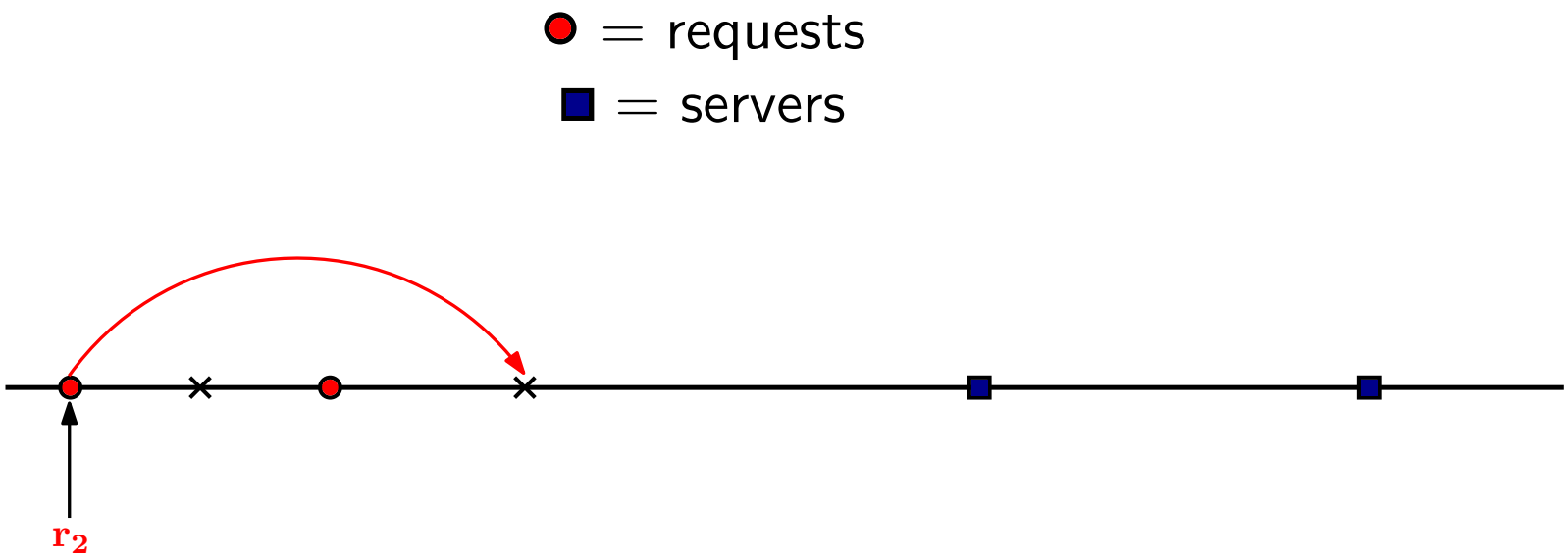

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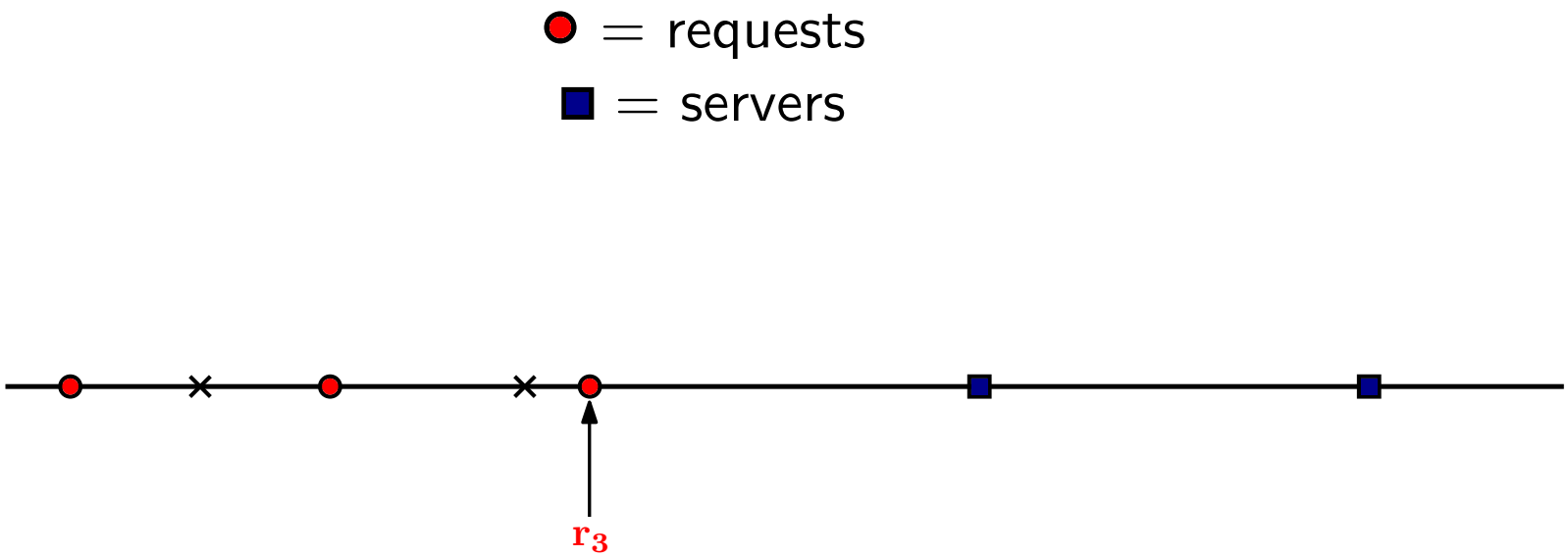

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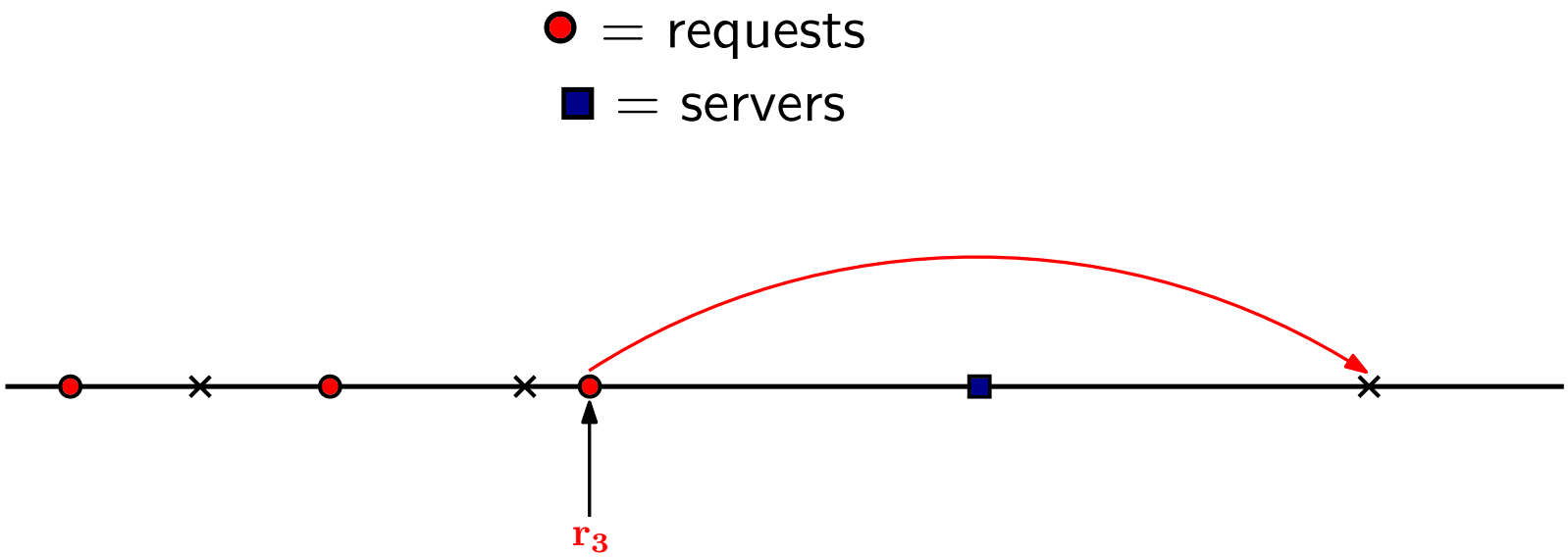

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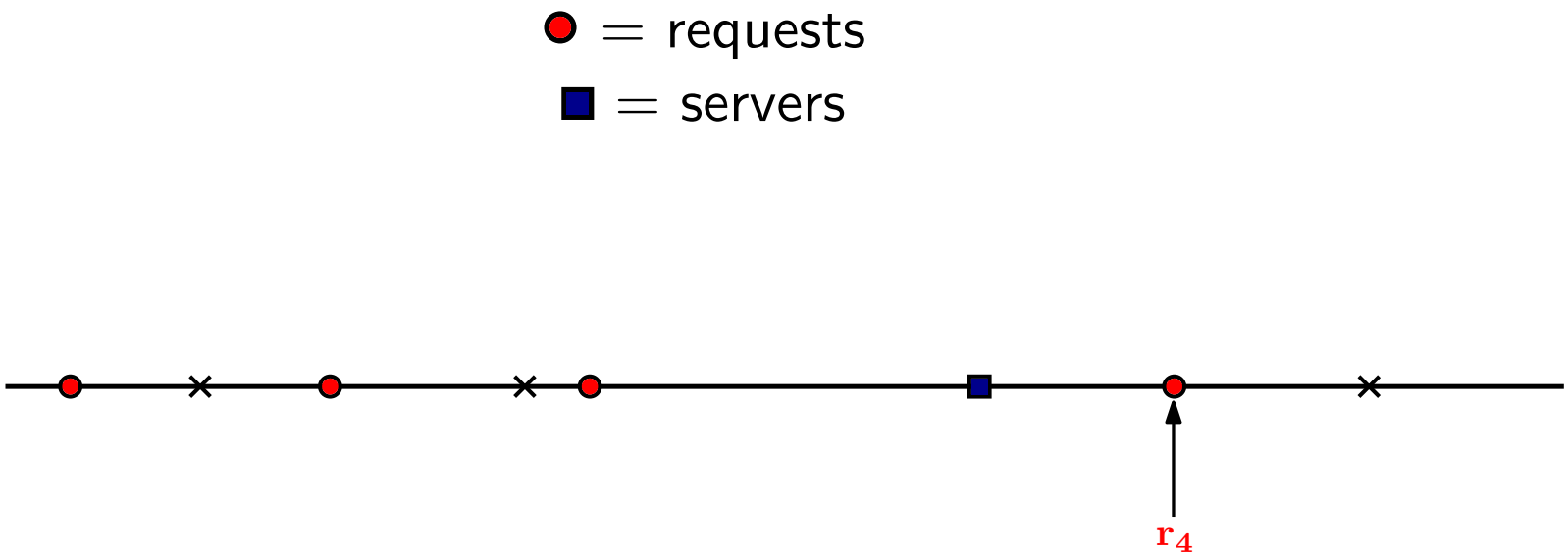

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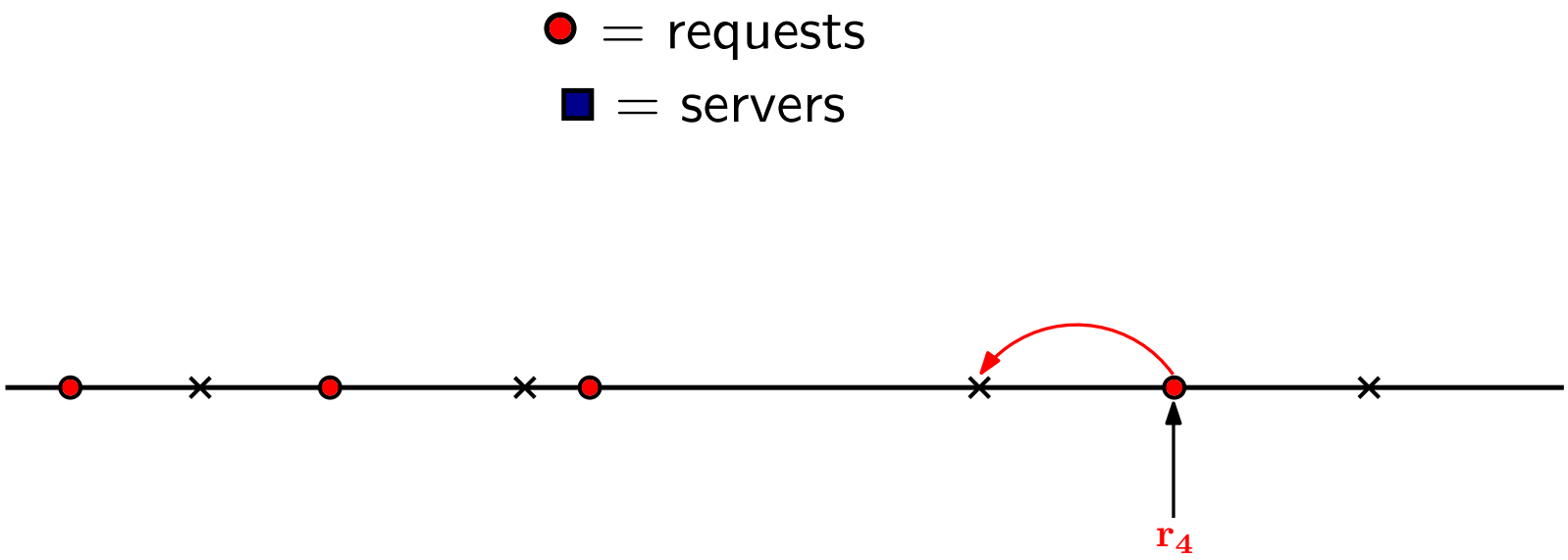

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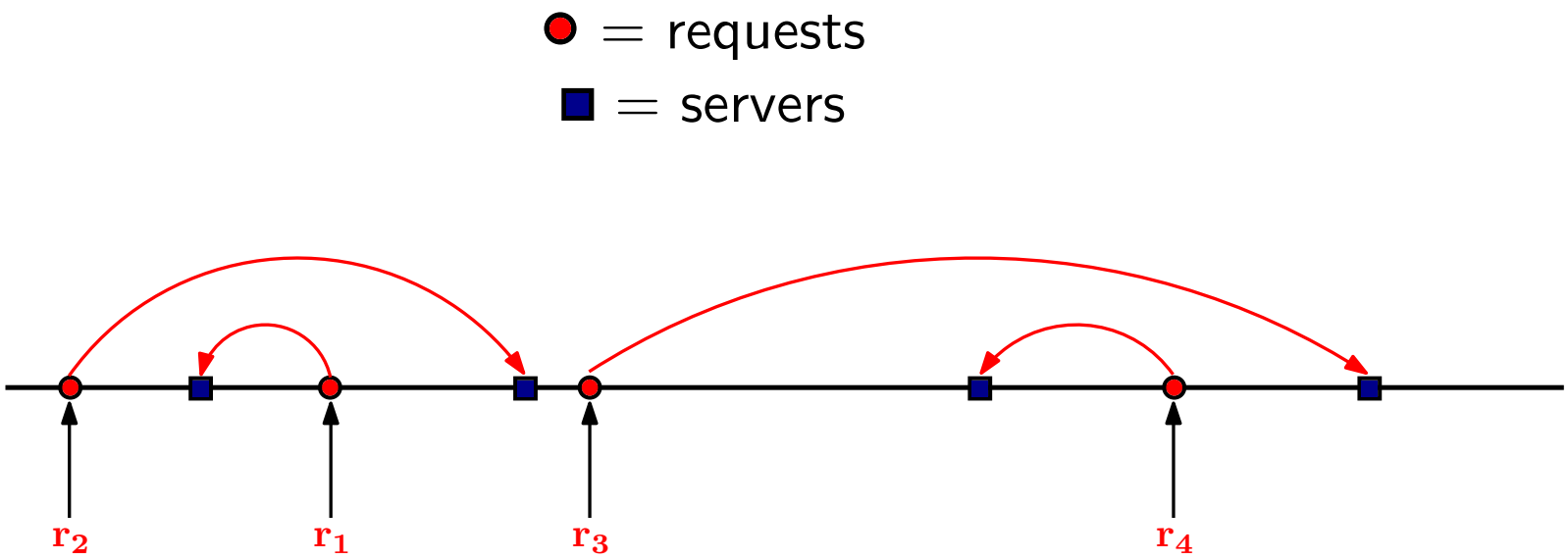

Total Cost: sum over all distances traveled in the matching

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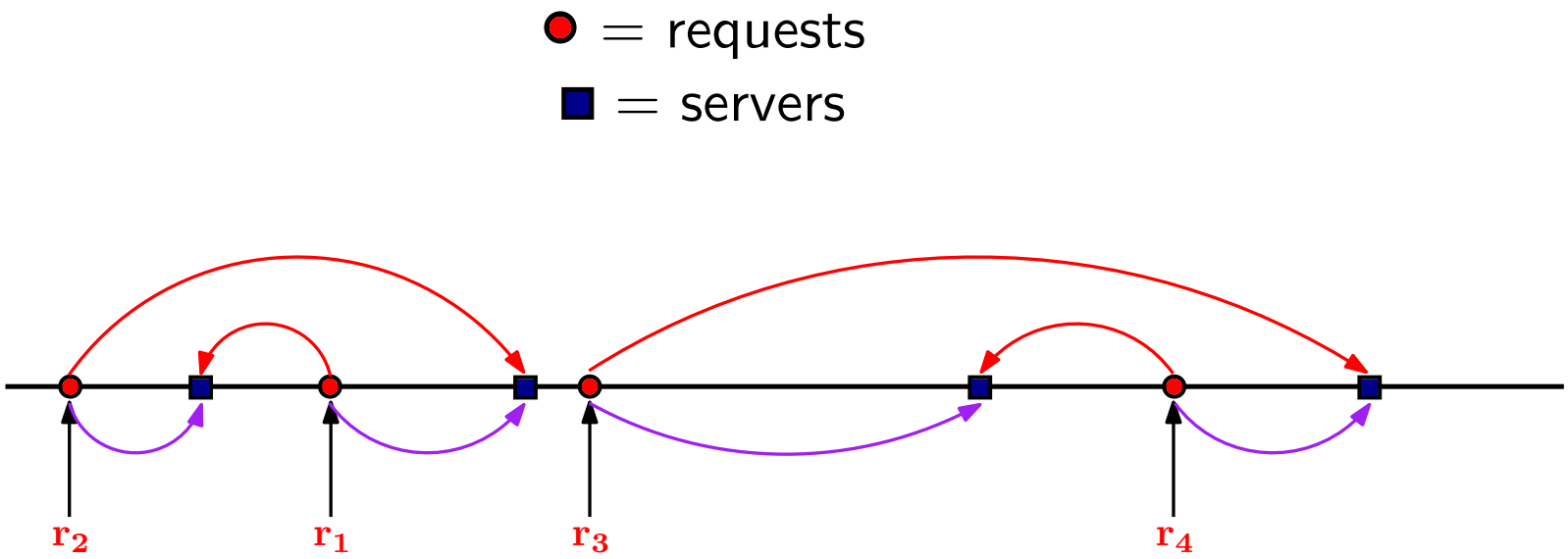

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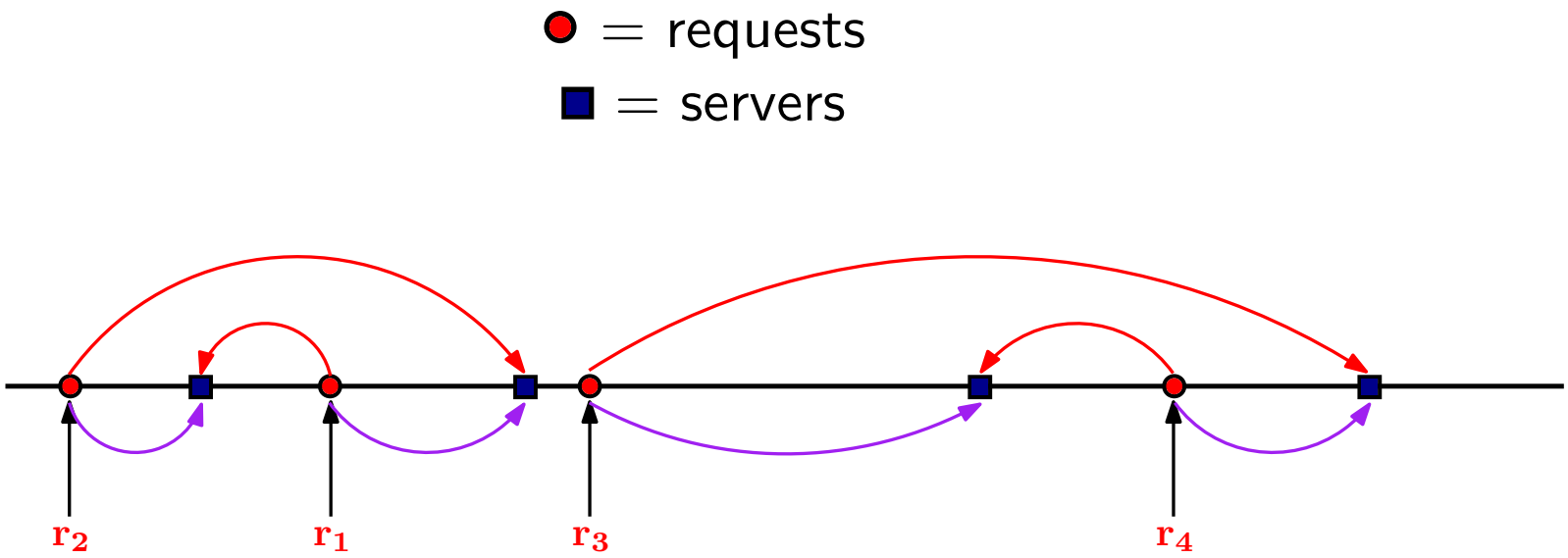

Competitive Ratio: how much we travel / how much OPT travels

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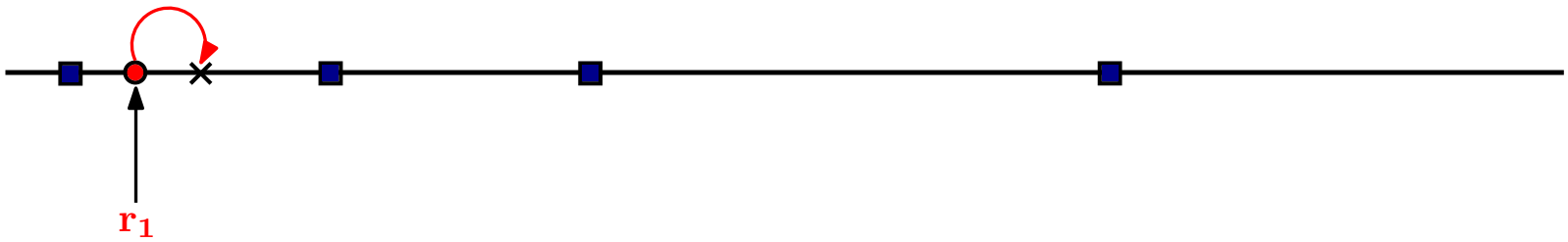

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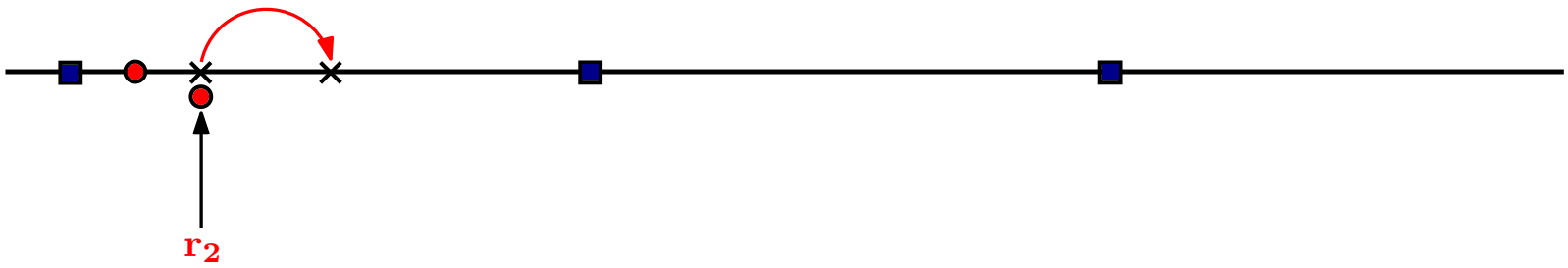

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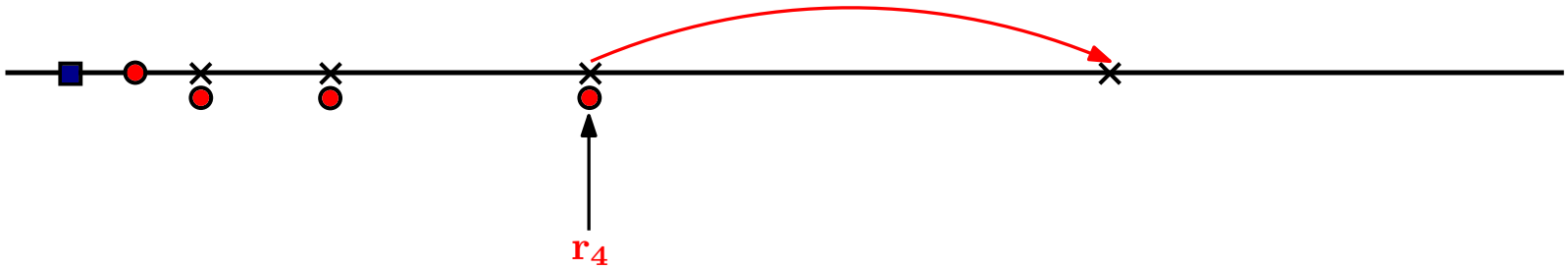

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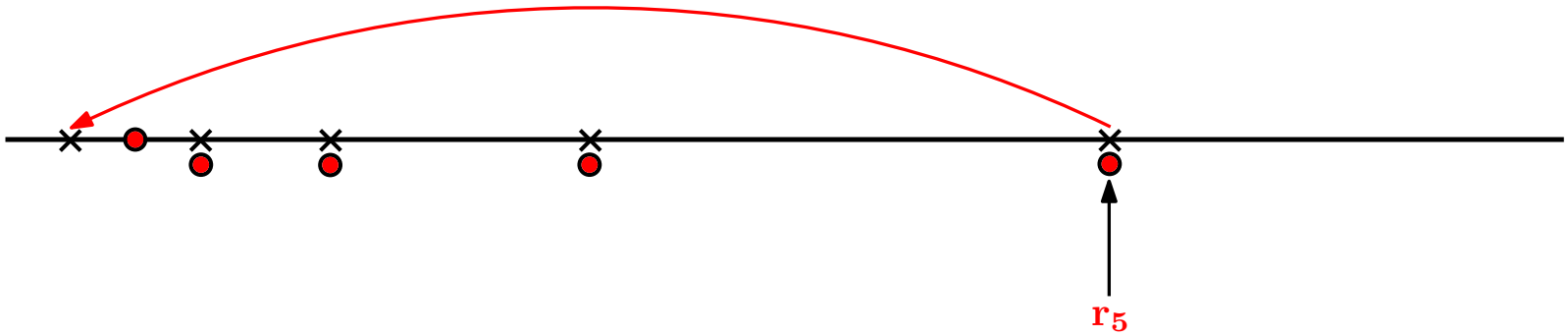

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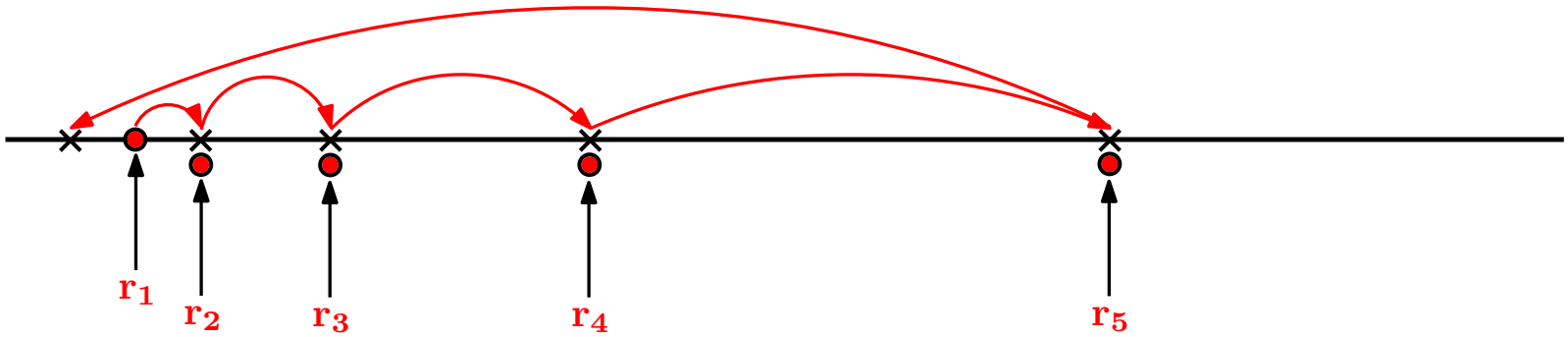

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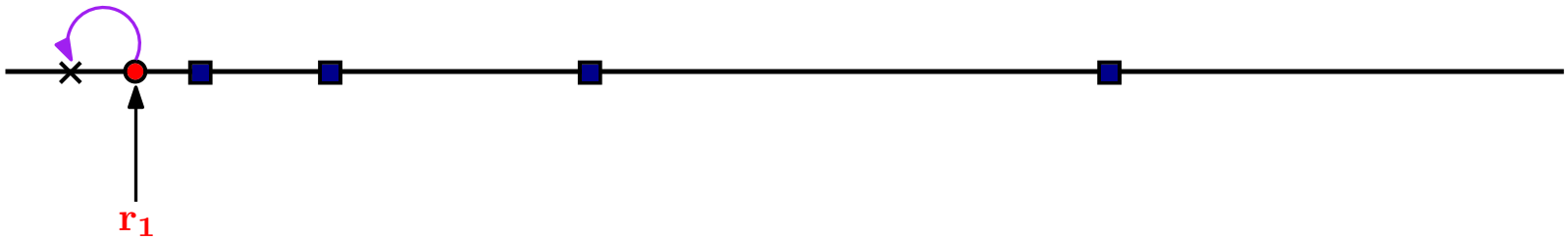

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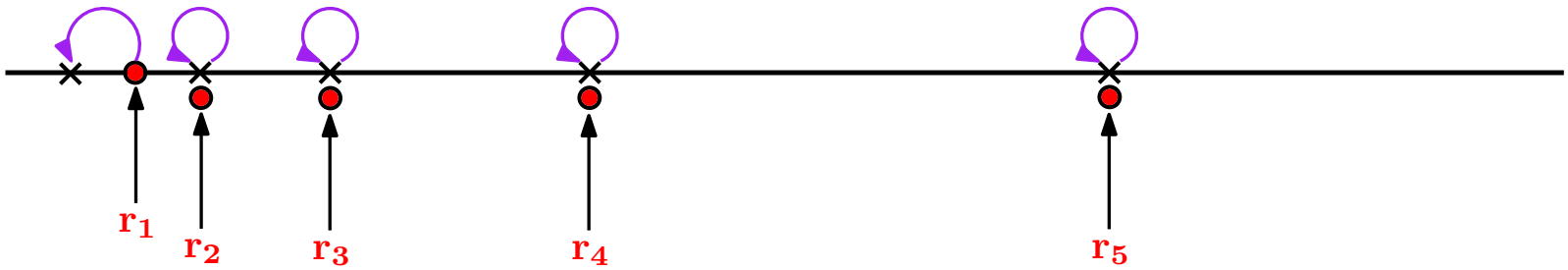

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Unfortunately, Greedy has competitive ratio $\Omega(2^k)$.

Outline

Bounds

The best deterministic upper bound for this problem is:

$$\text{competitive ratio} \leq 2k - 1$$

from [Kalyanasundaram, Pruhs 1993], and also [Khuller, Mitchell, V. Vazirani 1994]

Randomization

But the $2k - 1$ bound applies to deterministic algorithms. Can randomization help?

Tightest upper bound for randomized algorithms for this problem:

$$\text{competitive ratio} \leq O(\log^2 k)$$

by Bansal et al. in 2007.

Our Results

- For the line, we give a simple $O(\log k)$ competitive randomized algorithm.
- For metrics with constant doubling dimension (ex: $\mathbb{R}^2$, $\mathbb{R}^3$), we give an $O(\log k)$ competitive randomized algorithm.

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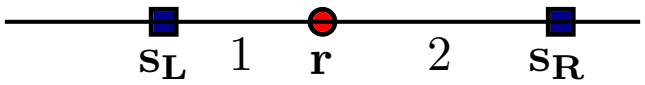

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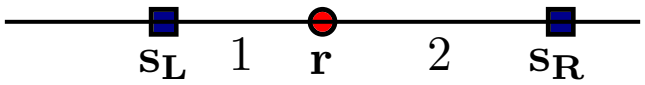

In this example, assign to s_L with probability:

$$\frac{d(r, s_R)}{d(s_L, s_R)} = \frac{2}{3}$$

Performance of Harmonic

- Define the aspect ratio of a metric space to be

$$\text{aspect ratio} = \frac{d_{max}}{d_{min}} = \frac{\max_{s,s'} d(s,s')}{\min_{s,s'} d(s,s')}$$

- We can show that Harmonic achieves a competitive ratio of $O(\log(\text{aspect ratio}))$

- Then, we can show that there is a preprocessing technique (a "guess-and-double" procedure) which allows us to assume that the line always has aspect ratio at most $O(k^3)$.

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$\Rightarrow$ We have an $O(\log k)$ competitive algorithm for the line

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Tree Metrics

- A tree metric is a metric space that can be "embedded" into a tree (all points lie at the leaves of the tree)
- Distances between points in the metric space correspond to the lengths of the paths between the points in the tree.
- There exists a way [FRT03] to transform any arbitrary metric space into a tree-based one, where distances are stretched by at most $O(\log k)$.

Our Techniques

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- Our techniques: transform metric into d-ary tree, then run $O(\log d)$ competitive algorithm on tree

Our Techniques

- This yields a competitive ratio of $O(\log k \log d)$ for arbitrary metric spaces.
- Metric spaces with doubling dimension d (like $\mathbb{R}^d$) can be converted into d-ary trees

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Gaps and Open Questions

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  Lower Bound: $\Omega(\log k)$
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- (Deterministic) Online Metric Matching for the Line:
  Lower Bound [BCR93]: 9
  Upper Bound [KP93]: $2k - 1$

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  Lower Bound: $\Omega(1)$
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- (Deterministic) Online Metric Matching for the Line:
  Lower Bound [F05]: $9 + \epsilon$
  Upper Bound [KP93]: $2k - 1$